

# The ADHM Construction, Hilbert Schemes, and the Heisenberg Algebra Lecture 2

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## Abstract

Continuing from the previous lecture, we consider the moduli space  $\mathcal{M}^{reg}(n, k)$  of framed “genuine” instantons and extend it to its compactification  $\hat{\mathcal{M}}(n, k)$  of so-called *ideal* instantons. This space has singularities, and we define its resolution  $\tilde{\mathcal{M}}(n, k)$  in terms of the original ADHM data. From here, we study the special case  $\tilde{\mathcal{M}}(1, k)$  as a Hilbert scheme of points on  $\mathbb{C}^2$ . This gives us a way to categorify Heisenberg algebras directly through the homology groups of these spaces.

## 1 Compactification and Resolution of $\mathcal{M}^{reg}(n, k)$

From before we have the ADHM construction:

**Definition 1.1** (ADHM datum). Two hermitian vector space  $V, W$  with  $\dim V = k, \dim W = n$  and maps  $B_1, B_2 \in \text{End}(V), I \in \text{Hom}(W, V), J \in \text{Hom}(V, W)$ . We have that the maps satisfy

$$[B_1, B_2] + IJ = 0 \tag{1}$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \tag{2}$$

Moreover  $U(V)$  acts on this space by

$$(gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \tag{3}$$

**Definition 1.2** ( $\mathcal{M}^{reg}(n, k)$ ). The moduli space of framed genuine instantons is given by taking the set of ADHM data  $[B_1, B_2, I, J]$  with trivial stabilizer under the action of  $U(V)$  and quotienting out by  $U(V)$ .

Equivalently, however, we can “forget” about the Hermitian structure on  $V$  (losing the notion of adjoints) and deal with a general vector space:

**Theorem 1.3.** *Let  $V, W$  be vector spaces of dimension  $\dim V = k, \dim W = n$ , and maps  $B_1, B_2 \in \text{End}(V), I \in \text{Hom}(W, V), J \in \text{Hom}(V, W)$ , satisfying only*

$$[B_1, B_2] + IJ = 0 \quad (4)$$

And  $\text{GL}(n, \mathbb{C})$  acts on this space in the same way as before, by:

$$(gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \quad (5)$$

Then the moduli space  $\mathcal{M}(n, k)$  of all such data with trivial  $\text{GL}_n$  stabilizer, quotiented by  $\text{GL}(n, \mathbb{C})$  is equivalent to  $\mathcal{M}^{fr}(n, k)$ .

This process of forgetting complex structure and then quotienting out by a larger group is similar to the identification:

$$\text{U}(n)/T_n \cong \text{GL}(n)/B_n \quad (6)$$

Where  $T_n$  is the toroidal subgroup of diagonal matrices and  $B_n$  is the Borel subgroup of upper-triangular matrices.

**Proposition 1.4.** *If ADHM datum  $[B_1, B_2, I, J]$  satisfies either*

1. (Stability) *There is no proper subspace  $S \subset V$  s.t.  $B_i(S) = S$  and  $I(W) \subset S$*
2. (Co-stability) *There is no proper subspace  $S \subset V$  s.t.  $B_i(S) = S$  and  $S \subset \ker J$*

*Then it has both nontrivial stabilizer and a closed orbit under  $\text{GL}(V)$ .*

*Proof.* Assume the stabilizer is nontrivial, so there is a  $g$  s.t.  $gI = I$ . Then  $\text{im } I \in \ker(g - 1_V) =: S$  and moreover since  $g^{-1}B_i g = B_i$ , we have that  $S$  is invariant under the  $B_i$  and contains  $\text{im } I$ . Similarly,  $Jg^{-1} = J \Rightarrow J(g - 1_V) = 0$  so  $\text{im}(g - 1_V) \in \ker J$ , violating co-stability as well. □

An ADHM datum that is both stable and co-stable is called **regular**.

For the hermitian case, it clear that  $(B_1, B_2, I, J)$  is stable iff  $(B_1^\dagger, B_2^\dagger, I^\dagger, J^\dagger)$  is co-stable. Equation (2) demands compatibility between these descriptions, so in fact it is enough to just require stability on the system, and co-stability follows:

$$\mathcal{M}^{reg}(n, k) = \{\text{Solutions to (1) + (2) + regularity}\}/\text{U}(V)$$

It is not too hard to see that all solutions of the ADHM equations giving rise to instantons are both stable and co-stable. As a result  $\mathcal{M}^{reg}(n, k)$  can be written as

$$\mathcal{M}^{reg}(n, k) = \{\text{Solutions to (1) + regularity}\}/\text{GL}(V)$$

The non-regular case is taken by compactifying this to allow for *all* ADHM data in the quotient:

**Definition 1.5.**  $\hat{\mathcal{M}}(n, k)$ , the compactification of  $\mathcal{M}^{reg}(n, k)$  is defined as

$$\hat{\mathcal{M}} := \{\text{Solutions to (1)}\}/\text{GL}(V)$$

This is a singular space in the sense that any Riemannian metric defined on the entire space will have a singularity at some point. For this reason, we define the **resolution** of this space.

**Definition 1.6** (Resolution). The resolution of a singular space  $X$  is a birational map  $\pi$ , together with a smooth variety,  $\tilde{X}$  such that  $\pi : \tilde{X} \rightarrow X$ .

**Definition 1.7.** We define  $\tilde{\mathcal{M}}(n, k)$  to be the set of data: satisfying

$$\hat{\mathcal{M}} := \{\text{Solutions to (1) + stability}\}/\text{GL}(V)$$

From the proof before, we see stability alone is enough to ensure a trivial stabilizer, so that this quotient is well-defined topologically.

Because stability on  $[B_1, B_2] + IJ$  is the same as co-stability on a corresponding dual system, it would have worked to also only have co-stability. Either definition works, so long as we have one stability condition but not the other.

**Theorem 1.8.**  $\tilde{\mathcal{M}}(n, k)$  is the minimal resolution of  $\hat{\mathcal{M}}(n, k)$  for all  $n, k$ .

**Observation 1.9.** Although  $\mathcal{M}^{reg}(1, k)$  is empty (for, as we know, there are no  $U(1)$  instantons), the definition of  $\tilde{\mathcal{M}}(n, k)$  gives rise to a nonempty set of solutions, as we shall see in the next section.

## 2 Hilbert Schemes of Points on $\mathbb{C}^2$

**Definition 2.1** (Hilbert Scheme). A Hilbert scheme  $\text{Hilb}_n X$  of  $n$  points on an algebraic variety  $X$  is given as the space of all ideals of codimension  $n$  in the  $\mathbb{C}[X]$ . That is, the set of ideals  $I$  so that  $\mathbb{C}[X]/I \cong V$  a vector space of dimension  $n$ .

This can be thought of as the moduli space of arrangements of  $n$  points on  $X$ , with subtleties when the points coincide. Grothendieck showed, through a much more general result, that this space is in fact a scheme.

**Example 2.2.** When  $\dim X = 1$  we have  $\text{Hilb}_n(X) \cong S^n X$  as the set of arrangements of  $n$  points modulo the symmetric group acting on these points by interchange.

**Example 2.3.** When  $X = \mathbb{C}$ , since  $\mathbb{C}[z]$  is a PID, we are interested in ideals  $I$  that are generated by a polynomial  $f$  of degree  $n$ . We are then looking all possible spaces  $V = \mathbb{C}[z]/I \cong \mathbb{C}^n$ .

Every such ideal gives rise to a map  $\varphi : \mathbb{C}[z]/I \rightarrow \mathbb{C}$  mapping

$$z \mapsto B \in \text{End}(V), \quad 1 \mapsto v_0 \in V \tag{7}$$

That is, we can represent multiplication by  $z$  as an operator  $B \circ V$  satisfying  $f(B) = 0$ .

The eigenvalues of  $B$  are exactly the points corresponding geometrically to this ideal (through nullstellensatz). We care about the coordinate-independent data:

$$(B, v_0)/GL(V) \tag{8}$$

Note that because of the existence of a cyclic vector, this system has the stability property: any space containing  $v_0$  and closed under the action of  $B$  is all of  $V$ . This constrains the Jordan form of  $B$ . We can go further and show that there is a 1-to-1 correspondence between such solutions and arrangements of  $n$  points.

In general the form of  $B$  will be

$$\begin{pmatrix} \lambda_1 & 1 & 0 & \dots & \dots \\ 0 & \lambda_1 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & \dots & \lambda_k & 1 \\ 0 & \dots & \dots & 0 & \lambda_k \end{pmatrix} \tag{9}$$

in jordan blocks, where  $\lambda_i \neq \lambda_j$  unless  $i = j$ . Then we can form a map from this data into  $S^n\mathbb{C}$  by  $B \rightarrow \sum_{\lambda \in \mathbb{C}} [\lambda] \lambda$ , where  $[\lambda]$  stands for the multiplicity. This is called the **Hilbert-Chow morphism**. For the case of  $\mathbb{C}$ , it is an isomorphism, as  $B$  can be recovered from the  $n$  points on  $\mathbb{C}$

We will be more interested in the case  $X = \mathbb{C}^2$ , the 2-dimensional complex space.

**Example 2.4.** As before, an element in the Hilbert scheme would correspond to an ideal  $I \in \mathbb{C}[x, y]$  s.t.  $V := \dim \mathbb{C}[x, y]/I = n$ . Consider the map

$$\varphi : \mathbb{C}[x, y] \rightarrow V \text{ s.t. } x \mapsto B_1, y \mapsto B_2, 1 \mapsto v_0 = I(1). \tag{10}$$

Then clearly we have  $[B_1, B_2] = 0$ , so we can write the  $B_k$  simultaneously in upper-triangular form as:

$$B_1 = \begin{pmatrix} \lambda_1 & \dots & \dots & \dots \\ 0 & \lambda_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}, B_2 = \begin{pmatrix} \mu_1 & \dots & \dots & \dots \\ 0 & \mu_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \mu_n \end{pmatrix} \tag{11}$$

Moreover we identify the vector  $v_0$  with the embedding operator  $I : \mathbb{C} \rightarrow V$  from a 1-dimensional space into  $V$ , so  $v_0 = I(1)$ .

We further have the Hilbert-Chow morphism to  $S^n\mathbb{C}^2$ :

$$\pi : (B_1, B_2, I) \rightarrow \{(\lambda_i, \mu_i)\}_{i=1}^n, \tag{12}$$

but now we no-longer preserve all information about the matrices, as the upper-triangular structure is unknown given the projection point.

Note that the orbit of  $v_0$  under  $B_1, B_2$  is all of  $V$ , so that any  $(B_1, B_2)$ -stable subspace of  $V$  containing  $v_0 = \text{im } I$  is all of  $V$ . This is exactly the stability condition from before. In fact, aside from  $[B_1, B_2] = 0$  instead of  $[B_1, B_2] + IJ = 0$ , this is exactly  $\tilde{\mathcal{M}}(1, n)$ . In fact, it is exactly this space:

**Theorem 2.5.** *We have an isomorphism of smooth spaces:*

$$\tilde{\mathcal{M}}(1, n) \cong \text{Hilb}_n(\mathbb{C}^2) \quad (13)$$

To prove this theorem, it is enough to just show  $J = 0$ , and we'll be done.

**Proposition 2.6.** *Consider the resolved space  $\tilde{\mathcal{M}}(1, k)$  from before (i.e. only stability), then any solution of ADHM has  $J = 0$ .*

*Proof.* Because stability implies that  $\mathbb{C}[B_1, B_2]I = V$ , it's enough to show  $Jp(B_1, B_2)I = 0$  for any monomial. We do this by induction on degree. For degree 0 we have  $J I = \text{Tr}(J I) = \text{Tr}(I J) = -\text{Tr}([B_1, B_2]) = 0$ .

For higher degree, we can use  $[B_1, B_2] = -J I$  to commute  $B_1, B_2$  across one another to get

$$\begin{aligned} J B_{\alpha_1} \dots B_2 B_1 \dots B_{\alpha_m} I &= J B_{\alpha_1} \dots I J \dots B_{\alpha_m} I + J B_{\alpha_1} \dots B_1 B_2 \dots B_{\alpha_m} I \\ &= J B_{\alpha_1} \dots B_1 B_2 \dots B_{\alpha_m} I \end{aligned} \quad (14)$$

so we can reduce this to  $J B_1^a B_2^b I$  and then use trace properties to get zero.  $\square$

This proves the theorem.

### 3 Homology Theory of $\text{Hilb}_n(\mathbb{C}^2)$

For a closed, oriented manifold  $X$  of dimension  $n$ , we have Poincare duality

$$H_i(X) \cong H^{n-i}(X) \quad (15)$$

When the manifold is not compact, we must pair the cohomology of *compactly-supported forms* with the homology.

$$H_i(X) \cong H_c^{n-i}(X) \quad (16)$$

and similarly, we define the **Borel-Moore homology** of locally finite chains by  $H_i^{lf} \cong H_c^{n-i}(X)$

**Definition 3.1.** The Borel-Moore (locally finite) homology  $H^{lf}$  is equivalent to the relative homology:

1.  $H_i^{lf}(X) := H_i(X \cup \{\infty\}, \{\infty\})$ , the one-point compactification of  $X$

This definition immediately yields

**Proposition 3.2.**  $H_{2m}^{lf}(\mathbb{C}^m) = \mathbb{Z}$  and otherwise is equal to zero.

by recognizing  $\mathbb{C}^m \cup \{\infty\}$  as  $S^{2m}$ . Further:

**Proposition 3.3.** *For a space  $X$  that is a disjoint union of a finite number of open subspaces  $X_\alpha$ :  $H_i^{lf}(X) = \bigoplus_\alpha H_i^{lf}(X_\alpha)$*

**Proposition 3.4.** *The Hilbert scheme  $\text{Hilb}_n(\mathbb{C}^2)$  is a disjoint union of open spaces  $C_\mu$  indexed by the partitions of  $n$ . Moreover,  $C_\mu \cong \mathbb{C}^{n+\ell(\mu)}$ , where  $\ell$  is the number of parts in the partition  $\mu$ .*

*Sketch.* Consider the action of the torus  $T^2 = \langle (t, q) \rangle \curvearrowright \text{Hilb}_n(\mathbb{C}^2)$  by its action on an ideal element  $((t, q)f)(x, y) = f(t^{-1}x, q^{-1}y)$ . The fixed points of this are the ideals

$$I_\mu = (x^a y^b : (a, b) \notin \mu) \quad (17)$$

Where  $\mu$  is viewed as its corresponding Young tableau on the plane  $\mathbb{N}^2$ . We then have corresponding complement ideals:

$$B_\mu = (x^a y^b : (a, b) \in \mu) \quad (18)$$

This gives rise to open sets  $U_\mu$  that cover  $\text{Hilb}_n(\mathbb{C}^2)$  defined by

$$U_\mu := \{I \in \text{Hilb}_n(\mathbb{C}^2) : B_\mu \text{ spans } \mathbb{C}[x, y]/I\} \quad (19)$$

and the closed sub-cells of these open sets are defined by:

$$C_\mu := \{I \in \text{Hilb}_n(\mathbb{C}^2) : \lim_{t \rightarrow 0} \lim_{q \rightarrow 0} (t, q)I = I_\mu\} \quad (20)$$

This limiting process picks out exactly the greatest monomials with nonzero coefficients from all the polynomials of the ideal.  $\square$

**Remark.** This type of idea, of decomposing a space into its different orbits, is universally used in studying not just Hilbert schemes but also projective spaces, Grassmannians, flag varieties, and other such spaces.

**Corollary 3.5.**

$$H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2)) = \bigoplus_{\mu} [C_\mu] = \bigoplus_{\mu} \mathbb{C} \quad (21)$$

where  $[C_\mu]$  denotes the fundamental Borel-Moore class (the top homology ring), which in this case is exactly the only nonzero one.

**Observation 3.6.** The dimension of this space  $H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2))$  is  $p(n)$ , the number of partitions of  $n$ .

Note however, that unlike  $S^n(\mathbb{C}^2)$ , which can also be covered by cells indexed by the partition type,  $\mu$ ,  $\text{Hilb}_n(\mathbb{C}^2)$  has the property that it has **constant rank** for its tangent space, equal to  $2n$ .

**Proposition 3.7.** We have a graded dimension for the algebra of homology rings for all Hilbert schemes of points over  $\mathbb{C}^2$  of:

$$\text{gdim} \bigoplus_n H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2)) = \sum_n p(n)q^n = \prod_m \frac{1}{1 - q^m} \quad (22)$$

This is exactly the graded dimension of the space of symmetric polynomials  $S(p_1, p_2, \dots)$  where  $\deg p_i = i$ . This is exactly a representation of the Heisenberg algebra, suggesting a connection between these two objects beyond just an isomorphism as graded vector spaces.

**Optional:**

**Theorem 3.8** (From Fulton). *The Poincare polynomial*

$$P_t^{lf}(X) := \sum_{n \geq 0} t^n \dim H_n^{lf}(X) \quad (23)$$

is equal to

$$\sum_{\mu} t^{2n+2\ell(\mu)} \quad (24)$$

for  $X = \text{Hilb}_n(\mathbb{C}^2)$

**Corollary 3.9.** *We have an identity for the polynomial*

$$\sum_{n=1}^{\infty} q^n P_t^{lf}(H_n) = \prod_{m=1}^{\infty} \frac{1}{1 - t^{2m+2} q^m} \quad (25)$$

**Theorem 3.10.** *The homology group  $H_*[\text{Hilb}_n(\mathbb{C}^2)]$  vanishes on odd degrees and otherwise is torsion free, with betti number;*

$$b_{2i}(\text{Hilb}_n(\mathbb{C}^2)) = p(n, n - i) \quad (26)$$

where  $p(n, a)$  is the number of partitions of  $n$  into  $a$  parts.

**Corollary 3.11.** *The Hilbert polynomial*

$$P_t(X) := \sum_{n \geq 0} t^n b_n(X) \quad (27)$$

is equal to

$$\sum_{\mu} t^{2n-2\ell(\mu)} \quad (28)$$

**Corollary 3.12.** *We have an identity for the polynomial*

$$\sum_{n=1}^{\infty} q^n P_t(\text{Hilb}_n(\mathbb{C}^2)) = \prod_{m=1}^{\infty} \frac{1}{1 - t^{2m-2} q^m} \quad (29)$$

We have constructed  $H_*^{lf}$  and  $H_*$ , and in fact all the nontrivial topological information is contained in the zero fiber  $Z_n$  of  $\pi : \text{Hilb}_n(\mathbb{C}^2) \rightarrow S^n(\mathbb{C}^2)$ .

## 4 Hilbert Schemes and the Heisenberg Algebra

The Heisenberg Lie Algebra  $\mathfrak{h}$  is defined by generators  $p_i, q_i$  so that:

$$[q_j, p_i] = c_j \delta_{ij}, \quad c_j \in \mathbb{C}^\times \quad (30)$$

so that on  $S(p_1, p_2, \dots)$  the  $p_j$  act by multiplication and the  $q_j$  by derivation  $c_j \frac{\partial}{\partial p_j}$ . We can pair  $\bigoplus_j p_j$  and  $\bigoplus_i q_i$  as dual vector spaces by:

$$\langle q_j, p_i \rangle = c_j \delta_{ij} \quad (31)$$

and with this can define the dual space

$$S^* = S(q_1, q_2 \dots) \quad (32)$$

with  $\deg q_j = j$  as before, and satisfying:

$$\langle q_1^{n_1} q_2^{n_2} \dots, p_1^{m_1} p_2^{m_2} \dots \rangle = n_1! c_1^{n_1} \delta_{n_1, m_1} n_2! c_2^{n_2} \delta_{n_2, m_2} \dots \quad (33)$$

Of course we have multiplication on  $S$  and comultiplication on  $S^*$ . In fact, since the commutation relations give a bilinear pairing:

$$S^* \otimes S \rightarrow S \quad (34)$$

by interpreting  $c_i$  as an element in  $S$ , we get

$$1 \otimes p_j = p_j, \quad q_i \otimes p_j = c_i \delta_{ij}, \quad (35)$$

and this gives **comultiplication** of the Heisenberg algebra:

$$\Delta : p_i \mapsto 1 \otimes p_i + p_i \otimes 1 \quad (36)$$

together with our previous multiplication. This makes this into a **Hopf algebra**

## 5 Geometric Realization of the Heisenberg Algebra

Going back to topology: we note that on any Homology ring  $H_*X$  we have that a continuous map  $\varphi : X \rightarrow Y$  between manifolds  $X$  and  $Y$  descends to a map between chains on  $X$  to chains on  $Y$ , and thus gives a map  $\varphi_* : H_*X \rightarrow H_*Y$  called the **pushforward** on homology rings, just by considering the homology of the mapped chains.

Similarly, any map  $\varphi : X \rightarrow Y$  induces a contravariant map  $\varphi^* \omega = \omega \circ \varphi$  on forms over  $Y$  called the **pullback**.

Now assume that  $f : X \rightarrow Y$  is a **proper map** (i.e.) the inverse image of any compact subset is compact. Then the pushforward descends to the locally-finite homology rings:

$$f_* : H_*^{lf}(X) \rightarrow H_*^{lf}(Y) \quad (37)$$

By defining  $\hat{f} : \hat{X} \rightarrow \hat{Y}$  on the 1-point compactifications of  $X, Y$  s.t.  $\hat{f}(\infty) = \infty$ . Then by properness,  $f$  is continuous on this space, so we can define the pushforward  $\hat{f}_*$  and pass this to the relative homology to get  $f_*$ .

From this, we can associate Hopf algebra structure to

$$\mathbb{H}^{lf} = \bigoplus_n \text{Hilb}_n(\mathbb{C}^2) \quad (38)$$

by considering a pair of three nested Hilbert schemes<sup>1</sup>:

$$\begin{aligned} H_{m+n \geq n} &:= \{V \in H_{m+n}, V'' \in H_n, V' = V/V'' \in H_m\} \\ &\subset H_m \times H_n \times H_{m+n} \end{aligned} \quad (39)$$

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<sup>1</sup>There is an issue when points collide that  $V'$  is not guaranteed to have a well-defined vector  $v_0$ . This can be resolved by looking at the open sets of distinct points first, and then taking the closure of this constructed space.



where for shorthand  $H_n$  denotes  $\text{Hilb}_n(\mathbb{C}^2)$ . We then have projections onto the components as the following diagram illustrates:

$$\begin{array}{ccc} & H_{m+n \geq n} & \\ p' \times p'' \swarrow & & \searrow p \\ H_n \times H_m & & H_{n+m} \end{array}$$

The following lemma can be proven by checking the preimages of points in  $S_\mu \mathbb{C}^2$  and noting that the fibers of the Hilbert-Chow morphism  $\pi^{-1}(\mu_i[x_i])$  are compact:

**Lemma 5.1.** *These projections  $p', p'', p$  are proper.*

So the projections therefore give rise to pushforwards on the Borel-Moore homology. Moreover, by this fact together with Poincare duality, we can identify elements in the homology rings with elements in the cohomology rings that we can *pull back*. Thus, we have maps:

$$\mu : H_*^{lf}(H_n) \otimes H_*^{lf}(H_m) \rightarrow H_*^{lf}(H_{n+m}) \quad (40)$$

given exactly by mapping homology elements

$$c_1 \otimes c_2 \mapsto p_*((p' \times p'')^*(c_1 \otimes c_2)) \quad (41)$$

This mirrors how multiplication in the Heisenberg algebra respects the degree grading. Similarly we have

$$\Delta : H_*^{lf}(H_{m+n}) \rightarrow H_*^{lf}(H_n) \otimes H_*^{lf}(H_m) \quad (42)$$

by

$$c \mapsto (p' \times p'')_*(p^*(c)) \quad (43)$$

**Theorem 5.2.** *The operations  $\mu, \Delta$  defined as above give  $\mathbb{H}^{lf}$  the structure of a graded Hopf algebra.*

*Proof.* We have already seen that these operations respect the grading. The associativity and coassociativity conditions follow from the functoriality of pushforward and pullback in the following diagrams: □

and

$$\begin{array}{ccccc} & & H_{m+n+r \geq n+r \geq r} & & \\ & & \swarrow p' \times p'' & & \searrow p \\ & H_{m+n \geq n} \times H_r & & & H_{n+m+r \geq r} \\ & \swarrow p' \times p'' \times id & \searrow p \times id & & \swarrow p' \times p'' \\ H_m \times H_n \times H_r & & H_{m+n} \times H_r & & H_{m+n+r} \\ & & & & \searrow p \end{array}$$

$$\begin{array}{ccccc} & & H_{m+n+r \geq n+r \geq r} & & \\ & & \swarrow p' \times p'' & & \searrow p \\ & H_m \times H_{n+r \geq r} & & & H_{n+m+r \geq n+r} \\ & \swarrow id \times p' \times p'' & \searrow id \times p & & \swarrow p' \times p'' \\ H_m \times H_n \times H_r & & H_m \times H_{n+r} & & H_{m+n+r} \\ & & & & \searrow p \end{array}$$

We can in fact go further and define the fiber  $F_\mu = \pi^{-1}S_\mu(\mathbb{C}^2)$  of arrangements of  $n$  points on  $\mathbb{C}$  of partition type  $\mu$ . We have that  $[F_\mu]$  is in fact well-defined and that

$$[F_\mu] \in H_{2(n+\ell(\mu))}^{lf}(\text{Hilb}_n(\mathbb{C}^2)) \quad (44)$$

**Theorem 5.3.** *The  $[F_\mu]$  form a basis for  $H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2))$ . Picking  $[F_n] \in H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2))$  corresponding to the fiber class with all  $n$  points coincident gives a multiplication operator  $P_n : H_m \rightarrow H_{n+m}$  corresponding to  $p_m$ .*

By Poincare duality, there is the intersection pairing

$$\cap : H_* \times H_*^{lf} \rightarrow \mathbb{C} \quad (45)$$

so that  $H_*$  can in fact be made to correspond to the dual space  $S^*$  of derivations. We can obtain fundamental classes in the regular homology  $[E_\mu]$  in a similar way, and have them form a basis for  $H_*\text{Hilb}_n(\mathbb{C}^2)$ . The remaining relations for the Heisenberg algebra can be obtained through careful calculation.

## References

- [1] Igor B. Frenkel *Compiled Lecture Notes #15, on Hilbert Schemes and the Heisenberg Algebra* 2016.
- [2] Igor B. Frenkel and Marcos Jardim *Complex ADHM equations, sheaves on  $\mathbb{P}^3$*  2008.
- [3] Alexander Kirillov Jr. *Quiver Varieties and Quiver Representations: 2016: Graduate Studies in Mathematics*, AMS.
- [4] Hiraku Nakajima *Hilbert Schemes of Points on Surfaces* 1991: University Lecture Series, AMS.
- [5] Hiraku Nakajima *Heisenberg Algebra and Hilbert Schemes of Points on Projective Surfaces* 1995.